

ON INEQUALITY $|z^n - 1| \geq |z - 1|$

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ABSTRACT. We prove that $|z^n - 1| \geq |z - 1|$ for all complex z satisfying $|z - 1/2| \leq 1/2$ and all real $n \geq 1$.

1. INTRODUCTION

R. Spira proved that $|w^{n+1} - 1| \geq |w^n||w - 1|$ for all Gaussian integers $w \in \mathbb{C}$ such that $\Re w \geq 1$ and all positive integers n , see [1]. He posed a question if this is true for all complex w such that $\Re w \geq 1$, n is again a positive integer, see [2]. The answer is affirmative, see [3], page 140. If we set $w = 1/z$, then the inequality is transformed into the following form: $|z^n - 1| \geq |z - 1|$ for $|z - 1/2| \leq 1/2$ and $n \geq 1$ is an integer.

In this note we prove that this inequality is valid for all *real* $n \geq 1$.

2. THE MAIN RESULT

As noted in the Introduction, we prove the following result.

Theorem 1. *For any real $n \geq 1$ we have*

$$(1) \quad |z^n - 1| \geq |z - 1|, \quad |z - 1/2| \leq 1/2.$$

If $n > 1$ and $z \neq 0, 1$, then the inequality is strict.

Our proof of the Theorem is based on the following lemma which is also of independent interest.

Lemma 1. *If $n > 3$ then*

$$(2) \quad \cos^n x < 1 - \sin x, \quad \frac{2\pi}{n+1} \leq x < \frac{\pi}{2}.$$

Proof of Lemma. Taking logarithms we transform our inequality into equivalent form

$$(3) \quad \frac{n}{2} \ln(1 - \sin^2 x) < \ln(1 - \sin x), \quad \frac{2\pi}{n+1} \leq x < \frac{\pi}{2}.$$

Next, using power series expansion of $\ln(1 - t)$ we obtain an equivalent inequality

$$(4) \quad \frac{n}{2} \sum_{k=1}^{\infty} \frac{\sin^{2k} x}{k} > \sum_{k=1}^{\infty} \frac{\sin^k x}{k}, \quad \frac{2\pi}{n+1} \leq x < \frac{\pi}{2},$$

or, by rearranging terms,

$$(5) \quad \sum_{k=1}^{\infty} \left(\frac{(n-1)\sin^{2k} x}{2k} - \frac{\sin^{2k-1} x}{2k-1} \right) > 0, \quad \frac{2\pi}{n+1} \leq x < \frac{\pi}{2}.$$

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Clearly, it suffices to prove that each term in the sum is strictly positive, and since $\sin x > 0$ for the allowed range of values of x this is equivalent to the following inequality

$$(6) \quad \sin x > \frac{2k}{(n-1)(2k-1)}, \quad k \geq 1, \quad \frac{2\pi}{n+1} \leq x < \frac{\pi}{2}, \quad n > 3.$$

However, the maximum of the right hand side over k is attained for $k = 1$ and the minimum of the left hand side over x is attained for $x = 2\pi/(n+1)$, so it suffices to verify the inequality

$$(7) \quad \sin \frac{2\pi}{n+1} > \frac{2}{n-1}, \quad n > 3.$$

Since $\sin x$ is strictly concave for $0 \leq x \leq \pi/2$ we have $\sin x > 2x/\pi$ for $0 < x < \pi/2$, setting $x = 2\pi/(n+1)$ this gives

$$(8) \quad \sin \frac{2\pi}{n+1} > \frac{2}{\pi} \frac{2\pi}{n+1} > \frac{2}{n-1},$$

the last inequality relies on the assumption $n > 3$. This proves our Lemma. \square

Proof of Theorem. Since

$$(9) \quad f(z) = \frac{z-1}{z^n-1}$$

is analytic in a neighborhood of $K = \{z \in \mathbb{C} : |z - 1/2| \leq 1/2\}$ it suffices, by the Maximum Modulus Principle, to prove our inequality for $z \in \partial K = C$. Let $z = re^{i\phi} \in C$. Since the inequality is obvious for $z = 0$ and for $z = 1$ we can assume $0 < r < 1$. Since both sides of inequality are invariant under complex conjugation we can also assume $0 \leq \phi \leq \pi/2$.

If $n\phi \leq 2\pi - \phi$, then the inequality holds for elementary geometric reasons. Indeed, in that case the point z^n lies on the circle $\{w \in \mathbb{C} : |w| = r^n\}$ and outside the angle $S_\phi = \{w = \rho e^{i\theta} : \rho \geq 0, -\phi < \theta < \phi\}$. Since $r^n < r$ it is easily seen that $|z^n - 1| > |z - 1|$.

Therefore, we can assume that $2\pi/(n+1) < \phi < \pi/2$. Note that this implies $n > 3$. Clearly $|z - 1| = \sin \phi$, therefore we have to prove that z^n lies outside the circle $|z - 1| = \sin \phi$. In fact, we are going to prove a stronger assertion: z^n lies to the left of the line l given by $\Re z = 1 - \sin \phi$, which is tangent to the mentioned circle. This can be expressed analytically as $\Re z^n < 1 - \sin \phi$. Since $r = \cos \phi$ this can be written as $\cos^n \phi \cos n\phi < 1 - \sin \phi$. However, in view of Lemma this is immediate: $\cos^n \phi \cos n\phi \leq \cos^n \phi < 1 - \sin \phi$ and the proof is finished. \square

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